ON A GENERALIZATION OF PERFECT b-MATCHING

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Summary. The paper is concerned with the existence of non-negative or positive solutions to $Af = \beta$, where A is the vertex-edge incidence matrix of an undirected graph. The paper gives necessary and sufficient conditions for the existence of such a solution.

Keywords: β-non-negative and β-positive graphs, perfect b-matching, system of linear equations.

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1. INTRODUCTION AND DEFINITIONS

Let $G = [V(G), E(G)]$ be a connected non-directed graph without loops or multiple edges with n vertices denoted by $v_1, v_2, \ldots, v_n$, and let $\beta = (b_1, b_2, \ldots, b_n)$ be an n-dimensional vector of positive real numbers. The graph G is called *β-non-negative* or *β-positive* if there exists a non-negative or positive solution f to the system of linear equations

$$\sum_{e \in E(G)} \eta(v_i, e) \, . \, f(e) = b_i \quad \text{for} \quad i = 1, 2, \ldots, n \,,$$

where $\eta(v_i, e) = 1$ when the vertex v_i and the edge e are incident or 0 otherwise. In other terms, if there exist non-negative or positive edge labels such that the sum of labels incident to v_i is b_i for all $1 \leq i \leq n$.

The solution f is called a *β-non-negative* or *β-positive labelling* of G with the indexing vector β. We use this terminology in accordance with [6], where another characterization of β-positive graphs was given.

If we consider the vector β and the solution of non-negative integers our problem coincides with the problem known as *perfect b-matchings* (see the book [5, p. 271]).

In the special case when β is a stationary vector of integers, the β-positive graph has been called a *regularisable graph* in Berge's paper [1] (see also [5, p. 218]), or a *semimagic graph* in [2], [3] and [7].

The aim of this paper is to characterize all vectors β for which the given graph G is β-non-negative or β-positive, respectively. Tutte's characterization of *perfect 2-matching graphs* [5, p. 216] is a particular case of our Theorem 1.

We use the terminology of Grünbaum's book [4]. Under an *elementary vector* ε_{ij}

assigned to the edge (v_i, v_j) we understand an n-dimensional vector with the i-th and j-th coordinates equal to 1 and all others equal to 0. The set of all elementary vectors assigned to edges of $E(G)$ will be denoted by $\mathscr{A}_G$. We say that the subset of $E(G)$ is linearly independent if the set of assignment vectors is linearly independent. The edges of a factor F of G are linearly independent iff every connected component of F is a tree or has exactly one odd circuit. By the symbol $\mathscr{K}_G$ we denote the set of all admissible indexing vectors of the given graph G. Evidently, every vector of $\mathscr{K}_G$ is a linear combination of vectors of $\mathscr{A}_G$ with non-negative coefficients. This yields

Lemma 1. *$\mathscr{K}_G$ is a cone generated by vectors of $\mathscr{A}_G$ with the apex* $(0, 0, \ldots, 0)$.

2. RESULTS CONCERNING THE CONE $\mathscr{K}_G$

Lemma 2. *The dimension of $\mathscr{K}_G$ is n if G is a non-bipartite graph and $n - 1$ if G is a bipartite graph.*

Lemma 2 is similar to Theorem 1 of [3].

In view of Theorem 1 of [4, p. 31] and [5, p. 256] the following assertion is true:

Lemma 3. *If G is a non-bipartite graph, then $\mathscr{K}_G$ is the intersection of a finite family $\mathscr{H}$ of closed halfspaces.*

Let $H_1, H_2, \ldots, H_k$ be the boundaries of halfspaces of $\mathscr{H}$. Each of these hyperplanes is determined by the origin and $n - 1$ linearly independent vectors of $\mathscr{A}_G$. We denote by δ_i the normal vector of the hyperplane H_i, $i = 1, 2, \ldots, k$. Without loss of generality, we assume that for every index i, δ_i is a normal vector such that its first non-zero coordinate is 1 or -1 and for all $\beta \in \mathscr{K}_G$ the scalar product $\langle \beta . \delta_i \rangle$ is non-positive.

By the symbol $\mathscr{D}$ we denote the set $\{\delta_1, \delta_2, \ldots, \delta_k\}$ of all normal vectors considered.

Corollary 1. *$\mathscr{K}_G$ is the set of all n-dimensional vectors β such that $\langle \beta . \delta_i \rangle \leqq 0$ for $i = 1, 2, \ldots, k$.*

3. THE STRUCTURE OF VECTORS OF $\mathscr{D}$

Let H be a hyperplane of an arbitrary halfspace of $\mathscr{H}$ and let $\delta = (d_1, d_2, \ldots, d_n) \in \in \mathscr{D}$ be its normal vector.

We divide the vertices of G into three sets:

$$\text{if} \quad d_i > 0 \quad \text{then} \quad v_i \in S_1^\delta ,$$
$$\text{if} \quad d_i < 0 \quad \text{then} \quad v_i \in S_{-1}^\delta , \quad \text{and}$$
$$\text{if} \quad d_i = 0 \quad \text{then} \quad v_i \in S_0^\delta .$$

By G^δ we denote the factor of G consisting of all edges assigned to the elementary vectors forming the hyperplane H. The edges of the factor G^δ are linearly independent. Since the cardinality of $E(G^\delta)$ is $n - 1$, therefore exactly one component of G^δ is a tree T and each other component contains one odd circuit C.

Let M be a component of G^δ having one odd circuit C. The relation $\langle \delta \, . \, \varepsilon_{ij} \rangle = 0$ holds for all edges $(v_i, v_j) \in E(M)$ only if every vertex of the circuit C belongs to S_0^δ, and consequently every vertex of the component M belongs to S_0^δ, too. The non-zero coordinates of δ are associated only to vertices of T.

Lemma 4. *If the edge* $e = (v_i, v_j) \in E(G^\delta)$ *and the vertex* $v_i \in S_1^\delta$, *then* $v_j \in S_{-1}^\delta$.

The proof follows from the fact that if the edge $(v_i, v_j) \in E(G)$, then the assigned elementary vector $\varepsilon_{ij} \in \mathscr{K}_G$ and so $\langle \varepsilon_{i,j} \, . \, \delta \rangle = d_i + d_j \leqq 0$.

Lemma 5. *The coordinates of the vector* δ *are* 1 *or* -1 *or* 0.

Proof. The first non-zero coordinate of δ, $d_i = 1$ or -1 corresponds to the vertex v_i which belongs to the component T of G^δ which is a tree. We have $\langle \varepsilon_{ij} \, . \, \delta \rangle = 0$ for all edges of $E(T)$ and consequently, if the coordinate $d_i = -1$, then $d_j = 1$ or if $d_i = 1$, then $d_j = -1$. So all vertices of T can be divided into two independent sets V_1 and V_2 such that if $d_i = 1$ then $v_i \in V_1$ and if $d_j = -1$ then $v_j \in V_2$.

Corollary 2. *The set* S_1^δ *is independent in* $V(G)$ *and the set of the neighbour vertices* $\Gamma(S_1)$ *is equal to the set* S_{-1}^δ.

4. CHARACTERIZATION OF β-NON-NEGATIVE GRAPHS

Theorem 1. *Let G be a connected graph with n vertices* $v_1, v_2, \ldots, v_n$ *and let* $\beta = (b_1, b_2, \ldots, b_n)$ *be a vector of non-negative numbers. The graph G is β-non-negative if and only if*

$$(1) \qquad \sum_{v_i \in S} b_i \leqq \sum_{v_j \in \Gamma(S)} b_j \quad \textit{for all independent} \quad S \neq \emptyset \quad \textit{of} \quad G\,.$$

Proof. Since no two vertices of S are joined by an edge the necessity of condition (1) is evident.

Let G be a non-bipartite graph. The set S_1^δ is independent in $V(G)$ and $S_{-1}^\delta = \Gamma(S_1^\delta)$ and so the scalar product $\langle \beta \, . \, \delta \rangle$ satisfies

$$\langle \beta \, . \, \delta \rangle = \sum_{v_i \in S_1^\delta} b_i - \sum_{v_j \in \Gamma(S_1^\delta)} b_j \leqq 0$$

for all vectors of $\mathscr{D}$, i.e. the vector $\beta \in \mathscr{K}_G$.

Let G be a bipartite graph with the partition V_1, V_2 of the vertex set $V(G)$ and let $|V(G)| \geqq 3$ (otherwise it is trivial). Then (1) implies

$$(2) \qquad \sum_{v_i \in V_1} b_i = \sum_{v_j \in V_2} b_j\,.$$

Now we form a non-bipartite graph G' by adding to edges of G one new edge connecting two vertices v_i and v_j of V_1. The graph G' has a β-labelling f. Evidently $f(v_i, v_j) = 0$ and so f is a β-non-negative labelling of G.

5. CHARACTERIZATION OF β-POSITIVE GRAPHS

Using the previous Lemmas and Corollaries and Theorem 1 it is easy to prove our main results.

Theorem 2. *Let G be a non-bipartite connected graph with n vertices $v_1, v_2, \ldots, v_n$ and let $\beta = (b_1, b_2, \ldots, b_n)$ be a vector of positive real numbers. The graph G is β-positive if and only if*

(3) $$\sum_{v_i \in S} b_i < \sum_{v_j \in \Gamma(S)} b_j \quad \text{for all independent} \quad S \neq \emptyset \quad \text{of} \quad G .$$

Proof. For every independent S there exists at least one edge joining some vertex of $\Gamma(S)$ with a vertex $v \notin S$. Therefrom the necessity of (3) follows.

We define a new vector β' with the coordinates $b'_i = b_i - \mu \deg(v_i)$, $i = 1, 2, \ldots, n$, where

$$\mu = \tfrac{1}{2} \min \{ \sum_{v_j \in \Gamma(S)} b_j - \sum_{v_i \in S} b_i \colon S \neq \emptyset \text{ is an independent subset of } V(G)\} .$$

Theorem 1 implies that G is a β'-non-negative graph with the labelling f'. So G is a β-positive graph with a labelling f such that $f(e) = f'(e) + \mu$ for all edges.

Theorem 3. *Let G be a bipartite graph with a partition V_1, V_2 having n vertices, and let $\beta = (b_1, b_2, \ldots, b_n)$ be a vector of positive real numbers. The graph G is β-positive if and only if*

(4) $$\sum_{v_i \in V_1} b_i = \sum_{v_j \in V_2} b_j$$

and

(5) $$\sum_{v_i \in S} b_i < \sum_{v_j \in \Gamma(S)} b_j \quad \textit{for all independent} \quad S \neq \emptyset, \quad V_1, V_2 .$$

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Súhrn

O ZOVŠEOBECNENÍ ÚPLNÉHO *b*-SPÁRENIA

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Práca sa zaoberá existenciou nezáporných, resp. kladných riešení systému lineárnych rovníc $Af = \beta$, kde A je vrcholovo-hranová incidenčná matica neorientovaného grafu a β n-rozmerný vektor z reálnych čísel. V práci sú uvedené nutné a postačujúce podmienky pre existenciu takýchto riešení.

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